Projected Written Notes from the M325K LECTURE ON TUESDAY, MARCH 7, 2024, ON Sec 6.3: Algebraic Proofs and Disproving a False Identity.

CLASS #16

Test 2 is on Thursday, MARCH 21, 2024

Sections Potentially covered:

Sec 4.3

Thm (N1B) 1, 3, 3

Using Thm (N1B) 2 in a proof

Sec 4.4, 4.5, 4.6

Proof Structure and proof methods

5.1

Σ NOTATION

Sec 5.2, 5.3, 5.4

Mathematical Induction

STRONG MATH'L Induction

W.O.P.

6.1, 6.2, 6.3

Set Theory

Elemental proofs.

(Disproving False Identities)

Algebraic proofs

You will not need to write a whole proof using the Well-Ordering Principle

Be able to Cite by Name:

① Theorem 4.3.1: "For all positive integers a and b, if $a|b$, then $a \le b$.

② Set Identities # 1, 2, 3, 4, 4, 9, 12

A Review of Cartesian Products:

Suppose A_1, A_2, A_3, A_4 are sets.

The Cartesian Product $A_1 \times A_2 \times A_3 \times A_4$ is the set of all ordered 4-tuples (x_1, x_2, x_3, x_4) such that

$$x_1 \in A_1,\ x_2 \in A_2,\ x_3 \in A_3 \text{ and } x_4 \in A_4.$$

Ex: Let $A = \mathbb{Z}^{pos}$, $B = \mathbb{Z}^{NEG}$, $C = \{0\}$, $D = \{1, 2\}$

$(3, -2, 0, 1) \in A \times B \times C \times D$

$(10, -1, 0, 2) \in A \times B \times C \times D$

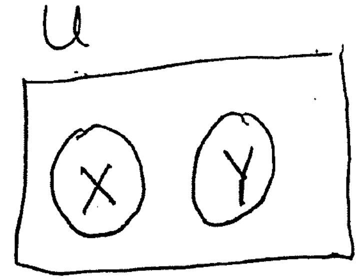
$(3, -2, 0, 4) \notin A \times B \times C \times D$ since $4 \notin \{0, 1\}$.

Ex: Let $B = \{1, 2\}$ and let $C = \{5\}$

$B \times B = \{ (1,1), (1,2), (2,1), (2,2) \}$

$$(B \times B) \times C = \{ ((1,1), 5), ((1,2), 5), ((2,1), 5), ((2,2), 5) \}$$

Def'n: Two Sets X and Y are Disjoint Sets if and only if $X \cap Y = \emptyset$.



Def'n: Let A be a set and let n be an integer with $n \geq 2$.

Let $A_1, A_2, \ldots, A_n$ be subsets of A.

The collection $\{A_1, A_2, \ldots, A_n\}$ of non-empty subsets of A is a partition of A if

(1) $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = A$

and (2) The subsets $A_1, A_2, \ldots, A_n$ are pair-wise disjoint, that is,

For all $i = 1, 2, \ldots, n$ and all $j = 1, 2, \ldots, n$ with $i \neq j$, $A_i \cap A_j = \emptyset$.

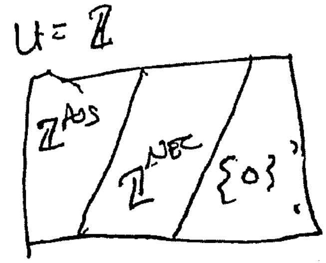
Ex: Let $A = \mathbb{Z}$. Let $A_1 = \mathbb{Z}^{POS}$, $A_2 = \mathbb{Z}^{NEG}$, $A_3 = \{0\}$.

$$\mathbb{Z}^{POS} \cup \mathbb{Z}^{NEG} \cup \{0\} = \mathbb{Z}$$

and $\mathbb{Z}^{POS} \cap \mathbb{Z}^{neg} = \emptyset$, $\mathbb{Z}^{POS} \cap \{0\} = \emptyset$,

$$\mathbb{Z}^{NEG} \cap \{0\} = \emptyset.$$

So, $\{\mathbb{Z}^{POS}, \mathbb{Z}^{NEG}, \{0\}\}$ is a PARTITION of $\mathbb{Z}$.



Some conjecture Identities are false identities.

Problem: Disprove the "identity"

"For all sets A, B and C,

$$A \cap (B \cup C) = (A \cap B) \cup C."$$

TO Prove: The Statement "For all sets A, B, and C,

$A \cap (B \cup C) = (A \cap B) \cup C$" is false.

[We need to prove the negation

"There exist sets A, B, and C such that

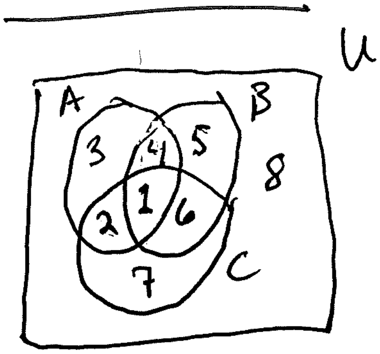
$A \cap (B \cup C) \neq (A \cap B) \cup C$".]

Proof:

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 5, 6\}$

$C = \{1, 2, 6, 7\}$

WORKSPACE



$B \cup C = \{1, 2, 4, 5, 6, 7\}$

$A \cap (B \cup C) = \{1, 2, 4\}$

$A \cap B = \{1, 4\}$, $(A \cap B) \cup C = \{1, 2, 4, 6, 7\}$

[We must prove that $A \cap (B \cup C) \neq (A \cap B) \cup C$]

The element 6 is such that $6 \in (A \cap B) \cup C$ and

$6 \notin A \cap (B \cup C)$. $\therefore (A \cap B) \cup C \not\subseteq A \cap (B \cup C)$.

$\therefore (A \cap B) \cup C \neq A \cap (B \cup C)$ by def of set equality.

$\therefore$ The Identity above is false by proof-by-counterexample.

QED.

Name: SOLUTIONS

In-Class Quiz No. 3 M 325K Spring 2024

1. For a set A, give the definition of "A $\neq \emptyset$, the empty set":

" Set A $\neq \emptyset$ " if and only if there exists an element $x \in \mathcal{U}$ such that $x \in A$.

2. For sets A and B, give the definition of " A $\subseteq$ B ":

" A $\subseteq$ B " if and only if, for all elements $x \in A$, $x \in B$.

3. Write an elemental proof of the given statement:

To Prove: For all sets A and B ,

if $x \in (A \cup B)$ and $x \in B^C$, then $x \in A$.

Proof: Let A and B be sets.

Suppose $x \in (A \cup B)$ and $x \in B^C$.

$\therefore$ $x \in A$ OR $x \in B$, by def'n of "UNION".

$\therefore$ Since $x \in B^C$, $x \notin B$ by def'n of "set complement".

$\therefore$ $x \in A$, by elimination.

$\therefore$ For all sets A and B, if $x \in (A \cup B)$ and $x \in B^C$, then $x \in A$, by Direct Proof

QED

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AN EXAMPLE OF AN ALGEBRIC PROOF IN SET THEORY

<u>TO PROVE:</u> FOR ALL SETS A, B, and C,

$$(A \cup B) - C = (A - C) \cup (B - C).$$

<u>PROOF:</u> Let A, B and C be any sets.

$\therefore (A \cup B) - C = (A \cup B) \cap C^c$
by the SET DIFFERENCE LAW,

$= C^c \cap (A \cup B)$, by the Commutative LAWS,

$= (C^c \cap A) \cup (C^c \cap B)$
by the DISTRIBUTIVE LAWS,

$= (A \cap C^c) \cup (B \cap C^c)$ by the Commutative Laws (applied twice),

$= (A - C) \cup (B - C)$
by the Set DIFFERENCE LAWS (APPLIED TWICE).

$\therefore (A \cup B) - C = (A - C) \cup (B - C)$ by the transitive property of "=".

$\therefore$ FOR ALL SETS A, B, and C,

$(A \cup B) - C = (A - C) \cup (B - C)$,
by DIRECT PROOF. Q.E.D.

Set Identities

An **identity** is an equation that is universally true for all elements in some set. For example, the equation $a + b = b + a$ is an identity for real numbers because it is true for all real numbers a and b. The collection of set properties in the next theorem consists entirely of set identities. That is, they are equations that are true for all sets in some universal set.

p.267

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

CITE BY NAME #1, 2, 3, 4, 6, 9, 12

from Class #24

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EXAMPLE 2:

TO PROVE: FOR ALL SETS A and B,

$$A \cap ((B \cup A^c) \cap B^c) = \emptyset.$$

PROOF: Let A and B be any sets.

$$A \cap ((B \cup A^c) \cap B^c) = A \cap (B^c \cap (B \cup A^c))$$

by the Commutative laws,

$$= (A \cap B^c) \cap (B \cup A^c)$$

by the Associative LAWS,

$$= ((A \cap B^c) \cap B) \cup ((A \cap B^c) \cap A^c)$$

by the Distributive Laws,

$$= (A \cap (B^c \cap B)) \cup ((B^c \cap A) \cap A^c)$$

by the Associative Laws AND the Commutative LAWS,

$$= (A \cap (B \cap B^c)) \cup (B^c \cap (A \cap A^c))$$

by the Commutative Laws AND the Associative Laws,

$$= (A \cap \emptyset) \cup (B^c \cap \emptyset)$$

by the Complement Laws (applied twice),

$$= \emptyset \cup \emptyset$$ by the UNIVERSAL BOUND LAWS (applied twice),

$$= \emptyset$$ by the IDENTITY LAWS.

$\therefore$ By transitivity, $A \cap ((B \cup A^c) \cap B^c) = \emptyset$.

$\therefore$ FOR ALL SETS A and B,

$A \cap ((B \cup A^c) \cap B^c) = \emptyset$, by DIRECT PROOF.

QED